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Comparison of three semi-analytical methods for solving (1 + 1)-dimensional dispersive long wave equations

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ABSTRACT

In this work, we consider how Adomian's decomposition method (ADM), the homotopy analysis method (HAM) and the homotopy perturbation method (HPM) can be used to investigate wave solutions of (1 + 1)-dimensional dispersive long wave equations. It is also worth noting that the advantage of the approximation of the series methodologies is a fast convergence of the solutions.

1. Introduction

In applied sciences, each physical event may be modeled mathematically. So, it is very important to have information about analytical solutions of the models because these solutions provide information about the character of the modeled event. Therefore, it is very important to find analytical solutions of linear or nonlinear ordinary and partial differential equations in physics, chemistry, biology and engineering areas. Recently, the scientists who study applied mathematics have focused on traveling wave solutions of nonlinear partial differential equations. Obtaining wave solutions has inspired scientists who study in applied fields as regards the structure of waves and mutual interactions. With such requirements, very effective methods for obtaining analytical solutions of nonlinear partial differential equations have been developed [1–15].

It is more difficult to obtain solutions of nonlinear partial differential equations than those of linear differential equations. Therefore, it may not always be possible to obtain analytical solutions of nonlinear partial differential equations. In this case, we use semi-analytical methods giving series solutions. In these kinds of methods, the solutions are sought in the form of series. Semi-analytical methods are based on finding the other terms of the series from given initial conditions for the problem being considered. At this point, we encounter the concept of convergence of the series. So, it is necessary to perform convergence analysis of these methods. As this convergence analysis can be carried out theoretically, one can gain information about the convergence of the series solution by looking at the absolute error between the numerical solution and the analytical solution. In some semi-analytic methods, a very good convergence can be achieved with only a few terms of the series, but more terms can be needed in some problems. That is, if the terms of the series increase, this provides better convergence to the analytical solution.

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The above-mentioned semi-analytical methods are frequently used by mathematicians. These methods include ADM, HAM and HPM. These methods have been successfully applied to solve linear and nonlinear differential equations [16–22]. When nonlinear differential equations are solved with semi-analytical methods, one may encounter some difficulties during the calculation because of the number of nonlinear terms and strongly nonlinear terms. So, when ADM is applied to nonlinear differential equations, Adomian's polynomials are used to overcome these difficulties. In same cases, it is necessary to make more calculations.

Many authors have studied such cases in their papers and give some implementations of equations by using ADM, HAM and HPM, and obtained some numerical solutions. Their results show the following. (i) Liao's HAM can produce approximations much better than the previous solutions for nonlinear differential equations. (ii) They have compared these three methods in order to obtain approximations and they observe that HAM is faster than HPM (see Figs. 1–6 of [23]). (iii) The other advantage of HAM is the flexibility in choosing an auxiliary parameter h to adjust and control the convergence and its rate for the solutions series, and for defining different functions which can originate from the nature of the problems considered [24]. (iv) If one compares the approximate solution with the exact solution when using HAM and HPM, one can see that according to the small parameter h, the convergence rate increases or decreases. (v) Chowdhury [25] shows that his numerical results obtained by using five-term HAM are exactly the same as the ADM solutions and HPM solutions for the special case of auxiliary parameter $h = -1$ and auxiliary function $H(x) = 1$. Because of this conclusion it was admitted that HPM and ADM are special cases of HAM.

In this study, we analyze ADM, HAM and HPM for (1 + 1)-dimensional dispersive long wave equations [26]. We will demonstrate that the approximate solutions of these equations are close to the corresponding exact solutions. Moreover, we show that ADM and HPM are special cases of HAM with $h = -1$ and $H(x) = 1$.

2. An analysis of the semi-analytical methods

(i) *The Adomian decomposition method*

The aim of the present section is to give an outline and implementation of ADM for nonlinear wave equations, to obtain analytic and approximate solutions which are obtained in a rapidly convergent series with elegantly computable components by this method. The approach is based on the choice of a suitable differential operator which may be ordinary or partial, linear or nonlinear, deterministic or stochastic [16–18]. It allows obtaining a decomposition series analytic solution of the equation which is calculated in the form of a convergent power series with easily computable components.

ADM is valid for ordinary and partial differential equations, no matter whether they contain small or large parameters, and thus is rather general. Moreover, the Adomian approximation series converge quickly. However, this method has some restrictions. Approximate solutions given by ADM often contain polynomials. In general, convergence regions of power series are small; thus acceleration techniques are often needed to enlarge convergence regions. This is mainly due to the fact that power series is often not an efficient set of base functions for approximating a nonlinear problem; but unfortunately ADM does not provide us with freedom to use different base functions.

We outline the method used here in order to approximate solutions by using ADM, considering the (1 + 1)-dimensional dispersive long wave equation with initial condition [27]

$$\begin{aligned} &u_t + uu_x + v_x = 0 \\ &v_t + u_x v + uv_x + \frac{1}{3}u_{xxx} = 0 \end{aligned} \tag{1}$$

$$\begin{aligned} u(x,0) &= -\sqrt{\frac{2k^2}{3}} - \frac{2k}{\sqrt{3}}\frac{1}{\sin(kx)}, \\ v(x,0) &= \frac{k^3}{3} - \frac{2k^2}{3}\frac{1}{\sin^2(kx)}, \end{aligned} \tag{2}$$

Eq. (1) can be written in an operator form as

$$\begin{aligned} &L_t u + N(u) + L_x v = 0 \\ &L_t v + M(u,v) + R(u,v) + \frac{1}{3}L_{xxx}u = 0, \end{aligned} \tag{3}$$

where $L_t = \frac{\partial}{\partial t}$, $L_{xxx} = \frac{\partial^3}{\partial x^3}$ and $N(u) = uu_x$, $M(u,v) = u_x v$, $R(u,v) = uv_x$. It is assumed that L_t^{-1} is an integral operator given by $L_t^{-1} = \int_0^t (.)\mathrm{d}t$. Operating with the integral operator L_t^{-1} on both sides of (3), we have

$$\begin{aligned} &L_t^{-1}(L_t u) + L_t^{-1}(N(u)) + L_t^{-1}(L_x v) = 0 \\ &L_t^{-1}(L_t v) + L_t^{-1}(M(u,v)) + L_t^{-1}(R(u,v)) + L_t^{-1}\left(\frac{1}{3}L_{xxx}u\right) = 0. \end{aligned} \tag{4}$$

Therefore, it follows that

$$\begin{aligned} u(x,t) &= u(x,0) - L_t^{-1}(N(u)) - L_t^{-1}(L_x v) \\ v(x,t) &= v(x,0) - L_t^{-1}(M(u,v)) - L_t^{-1}(R(u,v)) - L_t^{-1}\left(\frac{1}{3}L_{xxx}u\right) \end{aligned} \tag{5}$$

where $N(u) = uu_x = \sum_{n=0}^{\infty} A_n$, $M(u,v) = u_x v = \sum_{n=0}^{\infty} B_n$, $R(u,v) = uv_x = \sum_{n=0}^{\infty} C_n$ and the polynomials A_n, B_n, C_n are the so-called Adomian polynomials [16,17]. Polynomials A_n, B_n, C_n can be written as follows:

$$\begin{aligned} A_0 &= u_0(u_0)_x, \\ A_1 &= u_1(u_0)_x + u_0(u_1)_x, \\ A_2 &= u_2(u_0)_x + u_1(u_1)_x + u_0(u_2)_x, \\ A_3 &= u_3(u_0)_x + u_2(u_1)_x + u_1(u_2)_x + u_0(u_3)_x, \\ &\vdots \\ B_0 &= (u_0)_x v_0, \\ B_1 &= (u_0)_x v_1 + (u_1)_x v_0, \\ B_2 &= (u_0)_x v_2 + (u_1)_x v_1 + (u_2)_x v_0, \\ B_3 &= (u_0)_x v_3 + (u_1)_x v_2 + (u_2)_x v_1 + (u_3)_x v_0, \\ &\vdots \\ C_0 &= u_0(v_0)_x, \\ C_1 &= u_1(v_0)_x + u_0(v_1)_x, \\ C_2 &= u_2(v_0)_x + u_1(v_1)_x + u_0(v_2)_x, \\ C_3 &= u_3(v_0)_x + u_2(v_1)_x + u_1(v_2)_x + u_0(v_3)_x, \\ &\vdots \end{aligned}$$

and a recurrence relation for Eq. (5) can be written as follows:

$$\begin{cases} u_0 = u(x,0) \\ u_{n+1} = -L_t^{-1}(A_n) - L_t^{-1}(L_x v_n) \end{cases} \tag{6a}$$

$$\begin{cases} v_0 = v(x,0) \\ v_{n+1} = -L_t^{-1}(B_n) - L_t^{-1}(C_n) - L_t^{-1}\left(\frac{1}{3}L_{xxx}u_n\right), \quad n \geq 0 \end{cases} \tag{6b}$$

where the terms of the decomposition series are obtained from

$$\begin{aligned} u_1 &= L_t^{-1}(A_0) - L_t^{-1}(L_x v_0) \\ u_2 &= L_t^{-1}(A_1) - L_t^{-1}(L_x v_1) \\ u_3 &= L_t^{-1}(A_2) - L_t^{-1}(L_x v_2) \\ &\vdots \end{aligned} \tag{7a}$$

$$\begin{aligned} v_1 &= -L_t^{-1}(B_0) - L_t^{-1}(C_0) - L_t^{-1}\left(\frac{1}{3}L_{xxx}u_0\right) \\ v_2 &= -L_t^{-1}(B_1) - L_t^{-1}(C_1) - L_t^{-1}\left(\frac{1}{3}L_{xxx}u_1\right) \\ v_3 &= -L_t^{-1}(B_2) - L_t^{-1}(C_2) - L_t^{-1}\left(\frac{1}{3}L_{xxx}u_2\right) \\ &\vdots \end{aligned} \tag{7b}$$

and we can obtain $u_1, u_2, \ldots$ and $v_1, v_2, \ldots$ by using Eqs. (7a)–(7b) with Eq. (2) as follows:

$$\begin{aligned} u_0 &= -\sqrt{\frac{2k^2}{3}} - \frac{2k}{\sqrt{3}}\frac{1}{\sin(kx)}, \\ u_1 &= \frac{2}{3}\sqrt{2}(k^2)^{3/2} t \cot(kx)\csc(kx), \end{aligned}$$

$$u_2 = -\frac{1}{3\sqrt{3}}k^5t^2(3+\cos(2kx))\csc^3(kx),$$
$$u_3 = -\frac{1}{27\sqrt{2}}k^6\sqrt{k^2}t^3(23\cos(kx)+\cos(3kx))\csc^4(kx),$$
$$u_4 = -\frac{1}{216\sqrt{3}}k^9t^4(115+76\cos(2kx)+\cos(4kx))\csc^5(kx),$$
$$\vdots$$
$$v_0 = \frac{k^3}{3} - \frac{2k^2}{3}\frac{1}{\sin^2(kx)},$$
$$v_1 = \frac{4}{3}\sqrt{\frac{2}{3}}k^3\sqrt{k^2}t\cot(kx)\csc^2(kx),$$
$$v_2 = -\frac{4}{9}k^6t^2(2+\cos(2kx))\csc^4(kx),$$
$$v_3 = -\frac{4}{27}\sqrt{\frac{2}{3}}k^7\sqrt{k^2}t^3(11\cos(kx)+\cos(3kx))\csc^5(kx),$$
$$v_4 = -\frac{2}{81}k^{10}t^4(33+26\cos(2kx)+\cos(4kx))\csc^6(kx),$$
$$\vdots$$

which is defined by all terms that arise from the initial condition and from integrating the source term and decomposing the unknown functions $u(x,t)$, $v(x,t)$ as a sum of components defined by the decomposition series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \qquad v(x,t) = \sum_{n=0}^{\infty} v_n(x,t). \tag{8}$$

(ii) *The homotopy analysis method*

In this section, HAM [28,29] is considered. HAM has been constructed and successfully implemented to obtain approximate and numerical solutions for many kinds of nonlinear problems (see [30–32] and the references cited therein). A very nice explanation of the basic ideas of HAM and its relationships with other analytic techniques, and some of its applications in science and engineering, are given in Liao's book [28]. There are many methods grouped with HAM in the literature which are implemented to solve nonlinear problems. Most of these groups of methods are in principle based on Taylor series in an embedding parameter. If one could guess the initial function and auxiliary linear operator well then one could get a very good approximations in a few terms, especially for small values of the variable of the series.

For the purpose of illustration of using HAM [28], let us consider Eqs. (1)–(2) in operator form:

$$\begin{aligned} &L_t u + u u_x + v_x = 0 \\ &L_t v + u_x v + u v_x + \frac{1}{3}u_{xxx} = 0 \end{aligned} \tag{9}$$

where L is a linear operator: $L \equiv \frac{\partial}{\partial t}$. Eq. (9) can be written in a nonlinear operator form:

$$\begin{aligned} N[\phi(x,t;q)] &= \frac{\partial\phi(x,t;q)}{\partial t} + \phi(x,t;q)\frac{\partial\phi(x,t;q)}{\partial x} + \frac{\partial\varphi(x,t;q)}{\partial x} \\ N[\varphi(x,t;q)] &= \frac{\partial\varphi(x,t;q)}{\partial t} + \frac{\partial\phi(x,t;q)}{\partial x}\varphi(x,t;q) + \phi(x,t;q)\frac{\partial\varphi(x,t;q)}{\partial x} + \frac{1}{3}\frac{\partial^3\phi(x,t;q)}{\partial x^3}, \end{aligned} \tag{10}$$

where $q \in [0,1]$ is an embedding parameter and $\phi(x,t;q)$, $\varphi(x,t;q)$ are functions.

From $u(x,0) = U_0(x)$, $v(x,0) = V_0(x)$ $-\infty < x < \infty$ it is straightforward to express the solutions u, v using a set of base functions:

$$\{e_n(x)t^n, n \geq 0\},$$

where $e_n(x)$ as a coefficient is a function with respect to x. This provides us with the so-called Rule of Solution Expression.

Following Liao's method [28,29], let $u(x,0) = U_0(x)$, $v(x,0) = V_0(x)$ indicate the initial guess of the exact solutions u, v, where $h \neq 0$ is an auxiliary parameter and $H(x,t) \neq 0$ is an auxiliary function. A zero-order deformation equation is constructed as

$$\begin{aligned} &(1-q)L[\phi(x,t;q) - u_0(x,t)] = qhH(x,t)N[\phi(x,t;q)] \\ &(1-q)L[\varphi(x,t;q) - v_0(x,t)] = qhH(x,t)N[\varphi(x,t;q)], \end{aligned} \tag{11}$$

with the initial condition

$$\begin{aligned} \phi(x,0;q) &= U_0(x) \\ \varphi(x,0;q) &= V_0(x), \end{aligned} \tag{12}$$

and when $q = 0$ and 1 the above equation has the solution

$$\begin{aligned} \phi(x,t;0) &= u_0(x,t) \\ \varphi(x,t;0) &= v_0(x,t), \end{aligned} \tag{13}$$

and

$$\begin{aligned} \phi(x,t;1) &= u(x,t) \\ \varphi(x,t;1) &= v(x,t), \end{aligned} \tag{14}$$

respectively.

Assume that the auxiliary function $H(x, t)$ and the auxiliary parameter h are properly chosen such that $\phi(x, t; q)$ and $\varphi(x, t; q)$ can be expressed using Taylor series:

$$\begin{aligned} \phi(x,t;q) &= u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t) q^n \\ \varphi(x,t;q) &= v_0(x,t) + \sum_{n=1}^{\infty} v_n(x,t) q^n, \end{aligned} \tag{15}$$

where

$$\begin{aligned} u_n(x,t) &= \left. \frac{1}{n!} \frac{\partial^n \phi(x,t;q)}{\partial q^n} \right|_{q=0} \\ v_n(x,t) &= \left. \frac{1}{n!} \frac{\partial^n \varphi(x,t;q)}{\partial q^n} \right|_{q=0}, \end{aligned} \tag{16}$$

and, besides that, the above series is convergent at $q = 1$. Using (13) and (14) then yields

$$\begin{aligned} u(x,t) &= u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t) \\ v(x,t) &= v_0(x,t) + \sum_{n=1}^{\infty} v_n(x,t). \end{aligned} \tag{17}$$

For the sake of simplicity, define the vectors

$$\begin{aligned} \vec{u}_n(x,t) &= \{u_0(x,t), u_1(x,t), \ldots, u_n(x,t)\} \\ \vec{v}_n(x,t) &= \{v_0(x,t), v_1(x,t), \ldots, v_n(x,t)\}. \end{aligned} \tag{18}$$

Differentiating the zero-order deformation equation (11) n times with respect to the embedding parameter q, then setting $q = 0$, and finally dividing by $n!$, the nth-order deformation equation is written as

$$\begin{aligned} L\left[u_n(x,t) - \chi_n u_{n-1}(x,t)\right] &= hH(x,t) R_n\left[\vec{u}_{n-1}(x,t)\right] \\ L\left[v_n(x,t) - \chi_n v_{n-1}(x,t)\right] &= hH(x,t) R_n\left[\vec{v}_{n-1}(x,t)\right], \end{aligned} \tag{19}$$

where

$$\begin{aligned} R_n\left[\vec{u}_{n-1}(x,t)\right] &= \frac{1}{(n-1)!} \left. \left\{ \frac{\partial^{n-1}}{\partial q^{n-1}} N\left[\sum_{m=0}^{\infty} u_m(x,t) q^m\right] \right\} \right|_{q=0} \\ R_n\left[\vec{v}_{n-1}(x,t)\right] &= \frac{1}{(n-1)!} \left. \left\{ \frac{\partial^{n-1}}{\partial q^{n-1}} N\left[\sum_{m=0}^{\infty} v_m(x,t) q^m\right] \right\} \right|_{q=0}, \end{aligned}$$

or

$$\begin{aligned} R_n\left[u_{n-1}\right] &= \frac{\partial u_{n-1}}{\partial t} + \sum_{i=0}^{n-1} u_i \frac{\partial u_{n-1-i}}{\partial x} + \frac{\partial v_{n-1}}{\partial x^3} \\ R_n\left[v_{n-1}\right] &= \frac{\partial v_{n-1}}{\partial t} + \sum_{i=0}^{n-1} \left(v_i \frac{\partial u_{n-1-i}}{\partial x} + u_i \frac{\partial v_{n-1-i}}{\partial x} \right) + \frac{1}{3} \frac{\partial^3 u_{n-1}}{\partial x^3}, \end{aligned} \tag{20}$$

and

$$\chi_n = \begin{cases} 0, & n \le 1, \\ 1, & n > 1. \end{cases} \tag{21}$$

Therefore the nth-order approximations of $u(x,t)$, $v(x,t)$ are given by

$$\begin{aligned} u(x,t) &\approx u_0(x,t) + \sum_{m=1}^{n} u_m(x,t) \\ v(x,t) &\approx v_0(x,t) + \sum_{m=1}^{n} v_m(x,t). \end{aligned} \tag{22}$$

Substituting Eq. (1) with the initial value (2) into Eq. (19) with (20) and using Mathematica, the series solutions of Eq. (1) can be constructed as an approximate series solution (22) and the terms of series (22) can be obtained as follows:

$$\begin{aligned}
u_0 &= -\sqrt{\frac{2k^2}{3}} - \frac{2k}{\sqrt{3}}\frac{1}{\sin(kx)}, \\
u_t &= -\frac{2}{3}\sqrt{2h}(k^2)^{3/2} t\cot(kx)\csc(kx), \\
u_2 &= -\frac{1}{9}ht\csc^3(kx)(\sqrt{3}hk^5 t\cos(2x) + 3(\sqrt{3}hk^5 t + \sqrt{2}(1+h)(k^2)^{3/2}\sin(2x))), \\
u_3 &= -\frac{1}{54}ht\csc^4(kx)(\sqrt{2}(k^2)^{3/2}(9+18h+h^2(9+23k^4t^2))\cos(kx) \\
&\quad + \sqrt{2}(k^2)^{3/2}(-9-18h+h^2(-9+k^4t^2))\cos(3kx) + 6\sqrt{3}h(1+h)k^5 t(5\sin(kx)+\sin(3kx))), \\
u_4 &= -\frac{1}{684}ht\csc^5(kx)(270\sqrt{3}hk^5 + 540\sqrt{3}h^2k^5t + 270\sqrt{3}h^3k^5t + 115\sqrt{3}h^3k^9t^3 \\
&\quad + 4\sqrt{3}hk^5t(-54-108h+h^2(-54+19k^4t^2))\cos(2kx)\sqrt{3}hk^5t(-54-108h+h^2(-54+k^4t^2)) \\
&\quad \times \cos(4kx) + 108\sqrt{2}(k^2)^{3/2}\sin(2kx) + 324\sqrt{2}h(k^2)^{3/2}\sin(2kx) + 324\sqrt{2}h^2(k^2)^{3/2}\sin(2kx) \\
&\quad + 108\sqrt{2}h^3(k^2)^{3/2}\sin(2kx) + 396\sqrt{2}h^2k^6\sqrt{k^2}t^2\sin(2kx)396\sqrt{2}h^3k^6\sqrt{k^2}t^2\sin(2kx) \\
&\quad - 54\sqrt{2}(k^2)^{3/2}\sin(4kx) - 162\sqrt{2}h(k^2)^{3/2}\sin(4kx) - 162\sqrt{2}h^2(k^2)^{3/2}\sin(4kx) \\
&\quad - 54\sqrt{2}h^3(k^2)^{3/2}\sin(4kx) + 18\sqrt{2}h^2k^6\sqrt{k^2}t^2\sin(4kx) + 18\sqrt{2}h^3k^6\sqrt{k^2}\sin(4kx)), \\
&\vdots \\
v_0 &= \frac{k^3}{3} - \frac{2k^2}{3}\frac{1}{\sin^2(kx)}, \\
v_1 &= -\frac{4}{3}\sqrt{\frac{2}{3}}k^3h\sqrt{k^2}t\cot(kx)\csc^2(kx), \\
v_2 &= -\frac{2}{9}hk^3t\csc^4(kx)(4hk^3t + 2hk^3t\cos(2kx) + \sqrt{6}(1+h)\sqrt{k^2\sin(2kx)}), \\
v_3 &= h\left(-\frac{4}{3}\sqrt{\frac{2}{3}}h(1+h)k^3\sqrt{k^2}t\cot(kx)\csc^2(kx) - \frac{4}{9}h(1+2h)k^6t^2(2+\cos(2kx))\csc^4(2kx)\right. \\
&\quad \left. - \frac{4}{27}\sqrt{\frac{2}{3}}h^2k^7\sqrt{k^2}t^3(11\cos(kx)+\cos(3kx))\csc^5(kx)\right) - \frac{2}{9}hk^3t\csc^4(kx)(4hk^3t + 2hk^3t\cos(kx) \\
&\quad + \sqrt{6}(1+h)k^2\sin(2kx)), \\
v_4 &= \frac{2}{81}hk^3t\csc^4(kx)(-108hk^3t - 216h^2k^3t - 108h^3k^3t - 3\sqrt{6}h(1+h)\sqrt{k^2}(6+h(3+22k^4t^2) \\
&\quad - 3(2+h)\cos(2kx))\cot(kx) - 6\sqrt{6}h^2k^4\sqrt{k^2}t^2\cos(3kx)\csc(kx) - 6\sqrt{6}h^3k^4\sqrt{k^2}t^2 \\
&\quad \times \cos(3kx)\csc(kx) - 33h^3k^7t^3\csc^2(kx) - h^3k^7t^3\cos(4kx)\csc^2(kx) - 2hk^3t\cos(kx) \\
&\quad \times (27(1+h)^2 + 13h^2k^4t^2\csc^2(kx)) - 9\sqrt{6}\sqrt{k^2}\sin(2kx) - 9\sqrt{6}h\sqrt{k^2}\sin(2kx)), \\
&\vdots
\end{aligned}$$

(iii) *The homotopy perturbation method*

In this section, the homotopy perturbation method [19,33,34] is considered. Unlike other perturbation methods, this method does not need a small parameter in an equation. According to this method, a homotopy with an imbedding parameter $p \in [0, 1]$ is constructed and the imbedding parameter is considered as a "small parameter". Thus, this method is called the homotopy perturbation method. To illustrate this method, we consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{23}$$

with boundary condition

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \qquad r \in \Gamma, \tag{24}$$

where $A(u)$ is written as follows:

$$A(u) = L(u) + N(u). \tag{25}$$

A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, and Γ is the boundary of the domain Ω. The operator A can generally be divided into two parts L and N, where L is a linear operator and N is a nonlinear operator. Thus, Eq. (23) can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0. \tag{26}$$

By the homotopy technique, one constructs a homotopy $v(r, p) : \Omega \times [0, 1] \to \mathbb{R}$ which satisfies

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad p \in [0, 1],\ r \in \Omega \tag{27}$$

where $p \in [0, 1]$ is an embedding parameter, and u_0 is an initial approximation of Eq. (23), which satisfies the boundary conditions. Clearly, from Eq. (27) we have

$$H(v, 0) = L(v) - L(u_0) = 0, \tag{28}$$

$$H(v, 1) = A(v) - f(r) = 0, \tag{29}$$

and the process of changing p from zero to unity is just that of changing $v(r, p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation, and $L(v) - L(u_0)$, $A(v) - f(r)$ are called homotopic.

We consider v as follows:

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \cdots = \sum_{n=0}^{\infty} p^n v_n. \tag{30}$$

According to the homotopy perturbation method, the best approximation solution of Eq. (26) can be explained as a series of powers of p:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + \cdots = \sum_{n=0}^{\infty} v_n. \tag{31}$$

The convergence of the series (30) is given in [19,33]. Some results have been discussed in [35–39].

We outline the method used here in order to approximate solutions by using HPM; we consider Eq. (1) with initial condition (2) and we can construct a homotopy as follows:

$$\begin{aligned} &(1 - p)\left(\frac{\partial Y}{\partial t} - \frac{\partial u_0}{\partial t}\right) + p\left(\frac{\partial Y}{\partial t} + Y\frac{\partial Y}{\partial x} + \frac{\partial W}{\partial x}\right) = 0 \\ &(1 - p)\left(\frac{\partial W}{\partial t} - \frac{\partial v_0}{\partial t}\right) + p\left(\frac{\partial W}{\partial t} + \frac{\partial Y}{\partial x}W + Y\frac{\partial W}{\partial x} + \frac{1}{3}\frac{\partial^3 Y}{\partial x^3}\right) = 0, \end{aligned} \tag{32}$$

where $p \in [0, 1]$. We suppose that the solution of Eq. (32) has the form

$$Y = Y_0 + pY_1 + p^2 Y_2 + p^3 Y_3 + \cdots \tag{33}$$

$$W = W_0 + pW_1 + p^2 W_2 + p^3 W_3 + \cdots. \tag{34}$$

Then, substituting Eq. (33)–(34) into Eq. (32),

$$\begin{aligned} &\frac{\partial Y_0}{\partial t} + p\frac{\partial Y_1}{\partial t} + p^2\frac{\partial Y_2}{\partial t} + p^3\frac{\partial Y_3}{\partial t} - \frac{\partial u_0}{\partial t} + p\frac{\partial u_0}{\partial t} + pY_0\frac{\partial Y_0}{\partial x} + p^2 Y_0\frac{\partial Y_1}{\partial x} + p^3 Y_0\frac{\partial Y_2}{\partial x} \\ &\quad + p^4 Y_0\frac{\partial Y_3}{\partial x} + p^2 Y_1\frac{\partial Y_0}{\partial x} + p^3 Y_1\frac{\partial Y_1}{\partial x} + p^4 Y_1\frac{\partial Y_2}{\partial x} + p^3 Y_2\frac{\partial Y_0}{\partial x} + p^4 Y_2\frac{\partial Y_1}{\partial x} + p^4 Y_3\frac{\partial Y_0}{\partial x} \\ &\quad + p\frac{\partial W_0}{\partial x} + p^2\frac{\partial W_1}{\partial x} + p^3\frac{\partial W_2}{\partial x} + p^4\frac{\partial W_3}{\partial x} + \cdots = 0 \end{aligned}$$

$$\frac{\partial W_0}{\partial t}+p\frac{\partial W_1}{\partial t}+p^2\frac{\partial W_2}{\partial t}+p^3\frac{\partial W_3}{\partial t}-\frac{\partial v_0}{\partial t}+p\frac{\partial v_0}{\partial t}+pW_0\frac{\partial Y_0}{\partial x}+p^2W_1\frac{\partial Y_0}{\partial x}+p^3W_2\frac{\partial Y_0}{\partial x}$$
$$+p^4W_3\frac{\partial Y_0}{\partial x}+p^2W_0\frac{\partial Y_1}{\partial x}+p^3W_1\frac{\partial Y_1}{\partial x}+p^4W_2\frac{\partial Y_1}{\partial x}+p^3W_0\frac{\partial Y_2}{\partial x}+p^4W_1\frac{\partial Y_2}{\partial x}+p^4W_0\frac{\partial Y_3}{\partial x}$$
$$+pY_0\frac{\partial W_0}{\partial x}+p^2Y_0\frac{\partial W_1}{\partial x}+p^3Y_0\frac{\partial W_2}{\partial x}+p^4Y_0\frac{\partial W_3}{\partial x}+p^2Y_1\frac{\partial W_0}{\partial x}+p^3Y_1\frac{\partial W_1}{\partial x}+p^4Y_1\frac{\partial W_2}{\partial x}$$
$$+p^3Y_2\frac{\partial W_0}{\partial x}+p^4Y_2\frac{\partial W_1}{\partial x}+p^4Y_3\frac{\partial W_0}{\partial x}+\frac{1}{3}p\frac{\partial^3 Y_0}{\partial x^3}+\frac{1}{3}p^2\frac{\partial^3 Y_1}{\partial x^3}+\frac{1}{3}p^3\frac{\partial^3 Y_2}{\partial x^3}+\frac{1}{3}p^4\frac{\partial^3 Y_3}{\partial x^3}+\cdots=0$$

and equating the terms with same powers of p,

$$p^0\colon \frac{\partial Y_0}{\partial t}-\frac{\partial u_0}{\partial t}=0, \tag{35}$$

$$p^1\colon \frac{\partial Y_1}{\partial t}+\frac{\partial u_0}{\partial t}+Y_0\frac{\partial Y_0}{\partial x}+\frac{\partial W_0}{\partial x}=0, \tag{36}$$

$$p^2\colon \frac{\partial Y_2}{\partial t}+Y_0\frac{\partial Y_1}{\partial x}+Y_1\frac{\partial Y_0}{\partial x}+\frac{\partial W_1}{\partial x}=0, \tag{37}$$

$$p^3\colon \frac{\partial Y_3}{\partial t}+Y_0\frac{\partial Y_2}{\partial x}+Y_1\frac{\partial Y_1}{\partial x}+Y_2\frac{\partial Y_0}{\partial x}+\frac{\partial W_2}{\partial x}=0, \tag{38}$$

$$\vdots$$

$$p^0\colon \frac{\partial W_0}{\partial t}-\frac{\partial v_0}{\partial t}=0, \tag{39}$$

$$p^1\colon \frac{\partial W_1}{\partial t}+\frac{\partial v_0}{\partial t}+W_0\frac{\partial Y_0}{\partial x}+Y_0\frac{\partial W_0}{\partial x}+\frac{1}{3}\frac{\partial^3 Y_0}{\partial x^3}=0, \tag{40}$$

$$p^2: \frac{\partial W_2}{\partial t}+W_1\frac{\partial Y_0}{\partial x}+W_0\frac{\partial Y_1}{\partial x}+Y_0\frac{\partial W_1}{\partial x}+Y_1\frac{\partial W_0}{\partial x}+\frac{1}{3}\frac{\partial^3 Y_1}{\partial x^3}=0, \tag{41}$$

$$p^3: \frac{\partial W_3}{\partial t}+W_2\frac{\partial Y_0}{\partial x}+W_1\frac{\partial Y_1}{\partial x}+W_0\frac{\partial Y_2}{\partial x}+Y_0\frac{\partial W_2}{\partial x}+Y_1\frac{\partial W_1}{\partial x}+Y_2\frac{\partial W_0}{\partial x}+\frac{1}{3}\frac{\partial^3 Y_2}{\partial x^3}=0, \tag{42}$$

$$\vdots$$

and solving Eqs. (35)–(42),

$$Y_0=-\sqrt{\frac{2k^2}{3}}-\frac{2k}{\sqrt{3}}\frac{1}{\sin(kx)},$$
$$Y_1=\frac{2}{3}\sqrt{2}(k^2)^{3/2}t\cot(kx)\csc(kx),$$
$$Y_2=-\frac{1}{3\sqrt{3}}k^5t^2(3+\cos(2kx))\csc^3(kx),$$
$$Y_3=\frac{1}{27\sqrt{2}}k^6\sqrt{k^2}t^3(23\cos(kx)+\cos(3kx))\csc^4(kx),$$
$$Y_4=-\frac{1}{216\sqrt{3}}k^9t^4(115+76\cos(2kx)+\cos(4kx))\csc^5(kx),$$
$$\vdots$$
$$W_0=\frac{k^3}{3}-\frac{2k^2}{3}\frac{1}{\sin^2(kx)},$$
$$W_1=\frac{4}{3}\sqrt{\frac{2}{3}}k^3\sqrt{k^2}t\cot(kx)\csc^2(kx),$$
$$W_2=-\frac{4}{9}k^6t^2(2+\cos(2kx))\csc^4(kx),$$
$$W_3=\frac{4}{27}\sqrt{\frac{2}{3}}k^7\sqrt{k^2}t^3(11\cos(kx)+\cos(3kx))\csc^5(kx),$$

Table 1a
Absolute error of $u(x,t)$ using ADM for Eq. (1) ($k = 0.05$).

$u(x,t) - \phi_5(x,t)$					
t_i/x_i	0.1	0.2	0.3	0.4	0.5
0.1	1.2581×10^{-6}	3.87396×10^{-5}	2.83479×10^{-4}	1.15266×10^{-3}	3.39836×10^{-3}
0.2	2.0051×10^{-8}	6.29048×10^{-7}	4.68495×10^{-6}	1.93698×10^{-5}	5.8017×10^{-5}
0.3	1.77212×10^{-9}	5.59566×10^{-8}	4.19365×10^{-7}	1.74439×10^{-6}	5.25564×10^{-6}
0.4	3.16462×10^{-10}	1.00255×10^{-8}	7.53771×10^{-8}	3.14524×10^{-7}	9.5053×10^{-7}
0.5	8.3126×10^{-11}	2.63868×10^{-9}	1.98778×10^{-8}	8.31025×10^{-8}	2.51619×10^{-7}

Table 1b
Absolute error of $v(x,t)$ using ADM for Eq. (1) ($k = 0.05$).

$v(x,t) - \varphi_5(x,t)$					
t_i/x_i	0.1	0.2	0.3	0.4	0.5
0.1	4.32968×10^{-5}	1.32509×10^{-3}	9.64143×10^{-3}	3.8995×10^{-2}	1.14397×10^{-1}
0.2	3.46135×10^{-7}	1.08242×10^{-5}	8.03653×10^{-5}	3.31274×10^{-4}	9.89373×10^{-4}
0.3	2.04169×10^{-8}	6.43278×10^{-7}	4.81076×10^{-6}	1.99692×10^{-5}	6.00425×10^{-5}
0.4	2.73602×10^{-9}	8.65338×10^{-8}	6.49553×10^{-7}	2.70605×10^{-6}	8.16522×10^{-6}
0.5	5.7514×10^{-10}	1.82323×10^{-8}	1.37168×10^{-7}	5.72716×10^{-7}	1.73187×10^{-6}

$$W_4 = -\frac{2}{81}k^{10}t^4(33 + 26\cos(2kx) + \cos(4kx))\csc^6(kx),$$

$$\vdots$$

then we can write the series solution of Eq. (1) by using Eq. (33)–(34) for $p \to 1$.

3. Numerical experiments and their discussion

In this section, we show the three-dimensional figures for $u(x,t)$ and $v(x,t)$ obtained by using ADM, HAM and HPM. Moreover, we give data for the errors between the numerical solutions and analytical solutions of $(1+1)$-dimensional dispersive long wave equations in tables and graphs. Subsequently, by drawing the graph of the auxiliary h parameter, we determine the convergence range for approximate solutions of $(1+1)$-dimensional dispersive long wave equations by using HAM. By giving values to the auxiliary h parameter at points within the specified convergence range, we show the best approach for values of the auxiliary h parameter. For numerical comparison purposes, we consider $(1+1)$-dimensional dispersive long wave equations. The formulas for the numerical results for ADM, HAM and HPM (when $H(x,t) = 1$) are given as follows:

$$\lim_{n\to\infty} \psi_n = u(x,t) \quad \text{where } \psi_n(x,t) = \sum_{k=0}^{n} u_k(x,t),\ n \geq 0. \tag{43}$$

The numerical results obtained with ADM, HAM and HPM for $(1+1)$-dimensional dispersive long wave equations are shown in Figs. 1, 2 and Tables 1–3. As seen in Tables 1–3, the series of solutions $u(x,t)$ and $v(x,t)$ of $(1+1)$-dimensional dispersive long wave equations are very close to the analytical solution considering only six terms. As can be seen in Tables 1–3, the numerical values calculated with ADM, HAM and HPM are the same. However, this situation occurs when $h = -1$. That is, if one chooses $h = -1$ for considering the problem, the numerical results of ADM and HPM converge to the numerical results of HAM. As seen in Figs. 3–4, we obtain the convergence range of the series solution by drawing a figure showing the h parameter in the numerical solutions obtained with HAM for the best convergence. According to Figs. 3–4, the convergence range of the series solutions $u(x,t)$ and $v(x,t)$ is $-1.4 \leq h \leq -0.6$, approximately. It is seen in Tables 4 and 5 that the best approach to the analytical solution is at $h = -1$. With the choice of parameter h made in this way, one can see the advantage of HAM. Thus, the auxiliary parameter h plays an important role within the framework of HAM. In the light of these findings, we can say that ADM and HPM are in fact special cases of HAM, when $h = -1$. This situation is also shown in [40–44].

4. Conclusion and remarks

A clear conclusion can be drawn from the numerical results: that the ADM, HAM and HPM algorithms provide highly accurate numerical solutions for nonlinear partial differential equations. It is also worth noting that the advantage of the approximation of the series methodologies is a fast convergence of the solutions. The illustrations show the dependence of the rapid convergence on the character and behavior of the solutions just as for a closed form solution. Because of the auxiliary parameter h and auxiliary $H(x,t)$ function, HAM provides a better convergence than the other numerical methods.

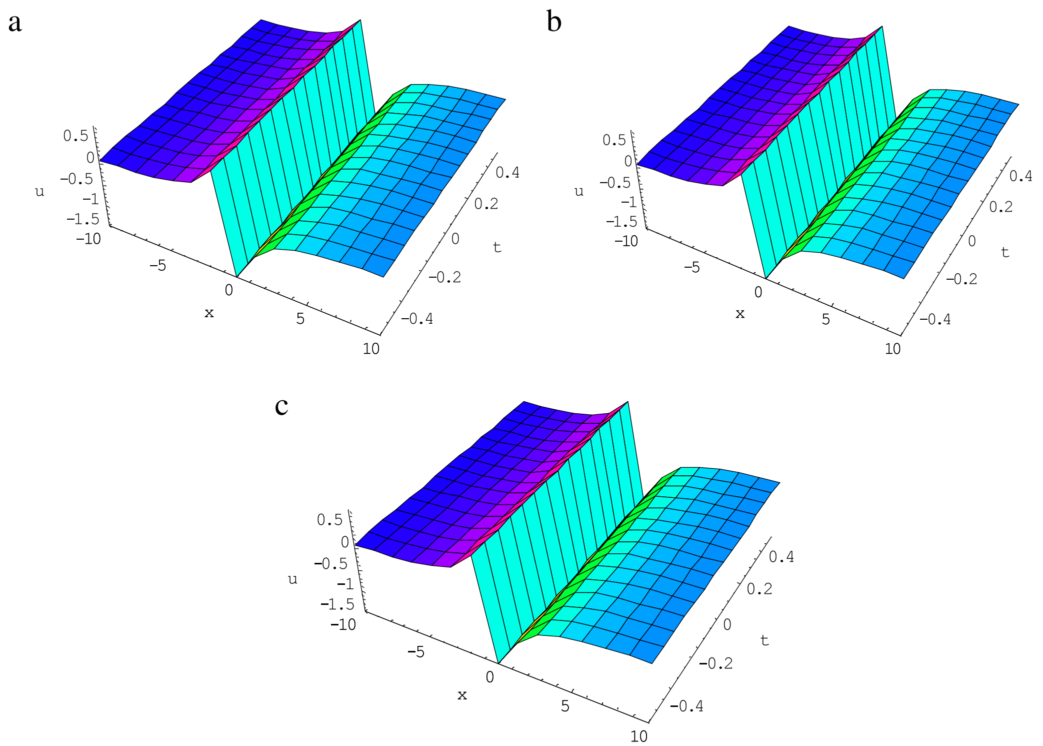

Fig. 1. Wave graphics of Eq. (1) for $u(x, t)$ (a) for ADM, (b) for HAM ($h = -1$), and (c) for HPM ($k = 0.05$).

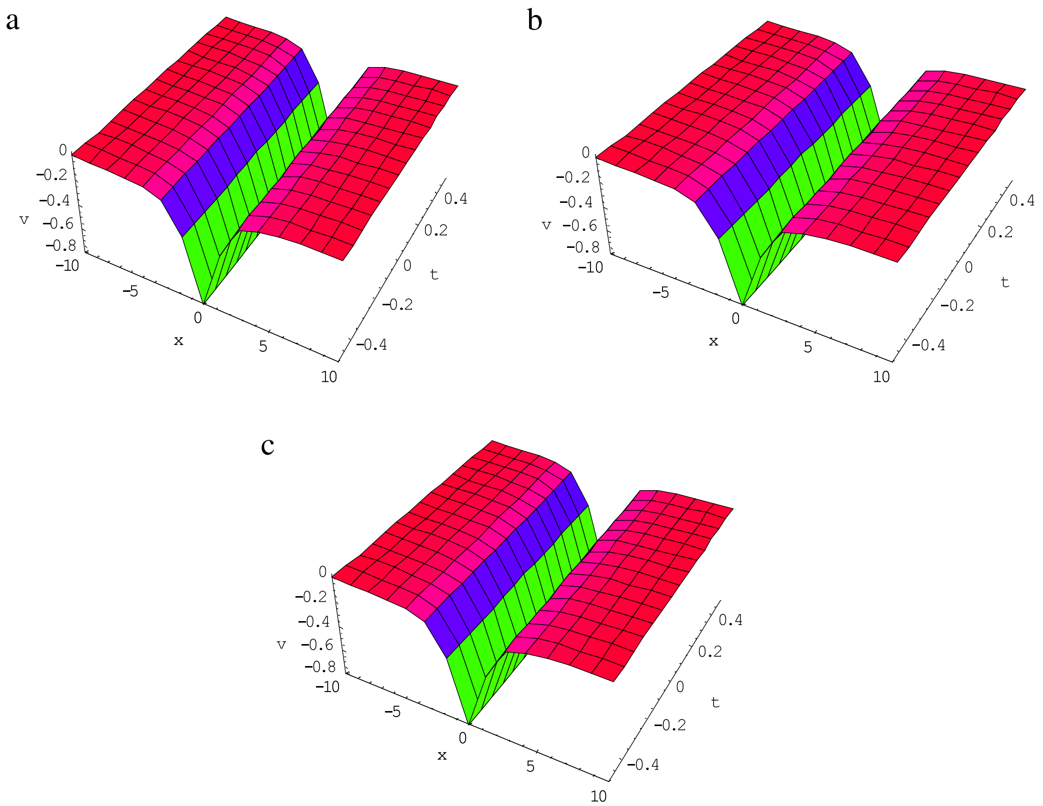

Fig. 2. Wave graphics of Eq. (1) for $v(x, t)$ (a) for ADM, (b) for HAM ($h = -1$), and (c) for HPM ($k = 0.05$).

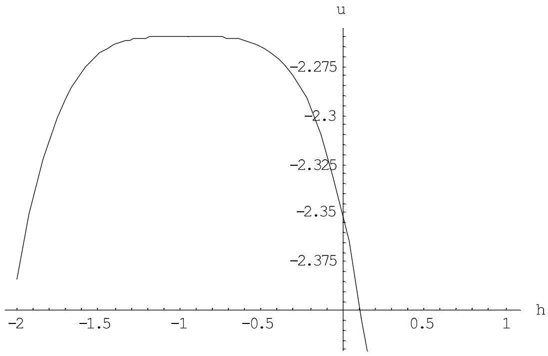

Fig. 3. Curve of arbitrary parameter h for $u(x, t)$.

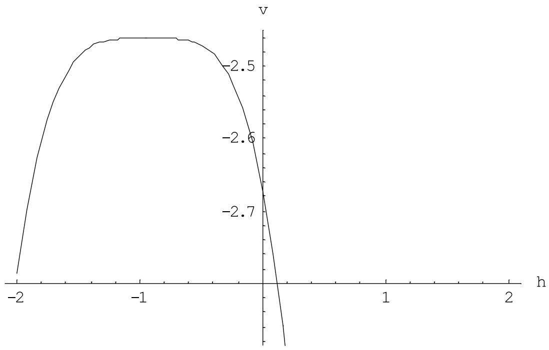

Fig. 4. Curve of arbitrary parameter h for $v(x, t)$.

Table 2a
Absolute error of $u(x, t)$ using HAM for Eq. (1) ($k = 0.05$, $h = -1$).

$u(x,t) - \phi_5(x,t)$					
t_i/x_i	0.1	0.2	0.3	0.4	0.5
0.1	1.2581×10^{-6}	3.87396×10^{-5}	2.83479×10^{-4}	1.15266×10^{-3}	3.39836×10^{-3}
0.2	2.0051×10^{-8}	6.29048×10^{-7}	4.68495×10^{-6}	1.93698×10^{-5}	5.8017×10^{-5}
0.3	1.77212×10^{-9}	5.59566×10^{-8}	4.19365×10^{-7}	1.74439×10^{-6}	5.25564×10^{-6}
0.4	3.16462×10^{-10}	1.00255×10^{-8}	7.53771×10^{-8}	3.14524×10^{-7}	9.5053×10^{-7}
0.5	8.3126×10^{-11}	2.63868×10^{-9}	1.98778×10^{-8}	8.31025×10^{-8}	2.51619×10^{-7}

Table 2b
Absolute error of $v(x, t)$ using HAM for Eq. (1) ($k = 0.05$, $h = -1$).

$v(x,t) - \varphi_5(x,t)$					
t_i/x_i	0.1	0.2	0.3	0.4	0.5
0.1	4.32968×10^{-5}	1.32509×10^{-3}	9.64143×10^{-3}	3.8995×10^{-2}	1.14397×10^{-1}
0.2	3.46135×10^{-7}	1.08242×10^{-5}	8.03653×10^{-5}	3.31274×10^{-4}	9.89373×10^{-4}
0.3	2.04169×10^{-8}	6.43278×10^{-7}	4.81076×10^{-6}	1.99692×10^{-5}	6.00425×10^{-5}
0.4	2.73602×10^{-9}	8.65338×10^{-8}	6.49553×10^{-7}	2.70605×10^{-6}	8.16522×10^{-6}
0.5	5.7514×10^{-10}	1.82323×10^{-8}	1.37168×10^{-7}	5.72716×10^{-7}	1.73187×10^{-6}

Table 3a
Absolute error of $u(x, t)$ using HPM for Eq. (1) ($k = 0.05$).

$u(x,t) - \phi_5(x,t)$					
t_i/x_i	0.1	0.2	0.3	0.4	0.5
0.1	1.2581×10^{-6}	3.87396×10^{-5}	2.83479×10^{-4}	1.15266×10^{-3}	3.39836×10^{-3}
0.2	2.0051×10^{-8}	6.29048×10^{-7}	4.68495×10^{-6}	1.93698×10^{-5}	5.8017×10^{-5}
0.3	1.77212×10^{-9}	5.59566×10^{-8}	4.19365×10^{-7}	1.74439×10^{-6}	5.25564×10^{-6}
0.4	3.16462×10^{-10}	1.00255×10^{-8}	7.53771×10^{-8}	3.14524×10^{-7}	9.5053×10^{-7}
0.5	8.3126×10^{-11}	2.63868×10^{-9}	1.98778×10^{-8}	8.31025×10^{-8}	2.51619×10^{-7}

Table 3b
Absolute error of $v(x, t)$ using HPM for Eq. (1) ($k = 0.05$).

$v(x,t) - \varphi_5(x,t)$					
t_i/x_i	0.1	0.2	0.3	0.4	0.5
0.1	4.32968×10^{-5}	1.32509×10^{-3}	9.64143×10^{-3}	3.8995×10^{-2}	1.14397×10^{-1}
0.2	3.46135×10^{-7}	1.08242×10^{-5}	8.03653×10^{-5}	3.31274×10^{-4}	9.89373×10^{-4}
0.3	2.04169×10^{-8}	6.43278×10^{-7}	4.81076×10^{-6}	1.99692×10^{-5}	6.00425×10^{-5}
0.4	2.73602×10^{-9}	8.65338×10^{-8}	6.49553×10^{-7}	2.70605×10^{-6}	8.16522×10^{-6}
0.5	5.7514×10^{-10}	1.82323×10^{-8}	1.37168×10^{-7}	5.72716×10^{-7}	1.73187×10^{-6}

Table 4
Absolute error of $u(x, t)$ using HAM for Eq. (1) at different values of arbitrary parameter h ($x = 0.5$, $k = 0.05$).

t	$h = -1.4$	$h = -1.2$	$h = -1$	$h = -0.8$	$h = -0.6$
0.1	4.32928×10^{-1}	7.67244×10^{-2}	3.39836×10^{-3}	3.53561×10^{-6}	1.16117×10^{-2}
0.2	4.64443×10^{-2}	5.782×10^{-3}	5.8017×10^{-5}	1.04884×10^{-4}	7.04149×10^{-3}
0.3	1.4752×10^{-2}	1.543×10^{-3}	5.25564×10^{-6}	1.10083×10^{-4}	4.08067×10^{-3}
0.4	6.92358×10^{-3}	6.52767×10^{-4}	9.5053×10^{-7}	8.99609×10^{-5}	2.60907×10^{-3}
0.5	3.95614×10^{-3}	3.48139×10^{-4}	2.51619×10^{-7}	7.10151×10^{-5}	1.80073×10^{-3}

Table 5
Absolute error of $v(x, t)$ using HAM for Eq. (1) at different values of arbitrary parameter h ($x = 0.5$, $k = 0.05$).

t	$h = -1.4$	$h = -1.2$	$h = -1$	$h = -0.8$	$h = -0.6$
0.1	8.74167	1.78628	1.14397×10^{1}	3.2739×10^{-4}	4.37402×10^{-3}
0.2	3.96884×10^{-1}	5.71937×10^{-2}	9.89373×10^{-4}	2.58064×10^{-4}	2.40746×10^{-2}
0.3	7.68156×10^{-2}	9.19333×10^{-3}	6.00425×10^{-5}	9.33438×10^{-5}	1.16364×10^{-2}
0.4	2.556×10^{-2}	2.72214×10^{-3}	8.16522×10^{-6}	1.202×10^{-4}	6.10085×10^{-3}
0.5	1.12423×10^{-2}	1.10465×10^{-3}	1.73187×10^{-6}	9.67905×10^{-5}	3.535×10^{-3}

When we take $h = -1$, ADM and HPM are derived from HAM. One of the other advantages of HAM is that the auxiliary linear operator L has been determined. There are many different studies of the comparison of semi-analytical methods in the literature [45–50].

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